

# Topologically protected colloidal transport above a square magnetic lattice

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We theoretically study the motion of magnetic colloidal particles above a magnetic pattern and compare the predictions with Brownian dynamics simulations. The pattern consists of alternating square domains of positive and negative magnetization. The colloidal motion is driven by periodic modulation loops of an external magnetic field. There exist loops that induce topologically protected colloidal transport between two different unit cells of the pattern. The transport is very robust against internal and external perturbations. Theory and simulations are in perfect agreement. Our theory is applicable to other systems with the same symmetry.

## I. INTRODUCTION

Controlling the transport of colloidal particles is a requisite in several applications such as lab-on-a-chip devices [1], drug delivery with colloidal carriers [2, 3], and computation with colloids [4].

Techniques to control the motion of colloids include the use of gradient fields [5], thermal ratchets [6–8], liquid crystal-based solvents [9, 10], and active particles [11]. Colloidal particles are usually polydisperse in e.g. size, mass, etc. Therefore, the transport of a collection of colloids using the above techniques results always in a dispersion of the motion. One can avoid this by using optical tweezers [12] but at the expenses of having to move the colloids on a one-by-one basis.

Topological protection is a promising tool to overcome these problems. If the dynamics depends only on a topological invariant it is possible to have total control over the colloidal motion, independently of the intrinsic characteristics of the particles. Recently, we have studied the motion of magnetic colloids above a hexagonal magnetic pattern [13]. The system is driven by an external magnetic field. The positions of the colloids above the pattern are given by the minima of the magnetic potential which has contributions from the static field of the pattern and the time dependent external field. The set of stationary points of the potential form a surface in the full phase space whose topological properties fully determine the colloidal motion. There exist transport modes that are topologically protected and therefore extremely robust against perturbations.

The topology of the stationary surface, and hence the topologically protected transport modes, are unique for each type of lattice. Here, we theoretically study the transport of diamagnetic colloidal particles above a square magnetic lattice, and compare the results with computer simulations.

## II. THEORY

The colloids move in a plane at a distance  $d > a$  above the pattern, with  $a$  the side-length of the unit cell of the pattern, see Fig. 1. A time-dependent external magnetic field  $\mathbf{H}_{\text{ext}}(t)$  drives the system. The variation in time of  $\mathbf{H}_{\text{ext}}(t)$  is slow enough such that the colloidal particles can adiabatically follow the minima of the magnetic potential at any time  $t$ . The magnetic potential is  $V = -\chi_{\text{eff}}\mu_0\mathbf{H} \cdot \mathbf{H}$ , where  $\mathbf{H}$  is the total magnetic field with contributions from the square pattern and the external potential,  $\chi_{\text{eff}} < 0$  is the effective magnetic susceptibility of the diamagnets in the solvent, and  $\mu_0$  is the vacuum permeability.

$\mathbf{H}$  can be expressed as a Fourier series with Fourier modes that decay exponentially with  $z$ . Hence, at high elevations,  $z > a$ , the potential is well approximated by  $V \propto \mathbf{H}_{\text{ext}}(t) \cdot \mathbf{H}_p(\mathbf{x}_A)$ , where

$$\mathbf{H}_p(\mathbf{x}_A) \propto \sum_{i=1}^4 \begin{pmatrix} q_{i,x} \sin(\mathbf{q}_i \cdot \mathbf{x}_A) \\ q_{i,y} \sin(\mathbf{q}_i \cdot \mathbf{x}_A) \\ q \cos(\mathbf{q}_i \cdot \mathbf{x}_A) \end{pmatrix}, \quad (1)$$

is, up to a multiplicative constant, the contribution from the magnetic pattern. Here,

$$\mathbf{q}_i = \frac{2\pi}{a} \begin{pmatrix} \sin(2\pi i/4) \\ -\cos(2\pi i/4) \end{pmatrix}, \quad i = 1, \dots, 4, \quad (2)$$

are the reciprocal lattice vectors of the second Brillouin zone with  $q = 2\pi/a$  their common magnitude.  $\mathbf{x}_A = x_1\mathbf{a}_1 + x_2\mathbf{a}_2$  with  $\mathbf{a}_i$  the basic lattice vectors of the square pattern (see Fig. 1), are the coordinates in *action space*  $\mathcal{A}$ , i.e., the plane above the pattern in which the colloids move. We vary  $\mathbf{H}_{\text{ext}}(t)$  on the surface of a sphere,

$$\mathbf{H}_{\text{ext}} = H_{\text{ext}}(\cos \phi_t \sin \theta_t, \sin \phi_t \sin \theta_t, \cos \theta_t). \quad (3)$$

The set  $(\theta_t, \phi_t)$  define our *control space*,  $\mathcal{C}$ , see Fig. 2a. We measure  $\theta_t$  with respect to the  $z$  axis and  $\phi_t$  with respect to  $\mathbf{a}_1$ . The system is driven with periodic closed loops of  $\mathbf{H}_{\text{ext}}(t)$ . There exist special loops that induce transport between different unit cells, i.e., when  $\mathbf{H}_{\text{ext}}$  returns to its initial position the particle is in a different unit cell.

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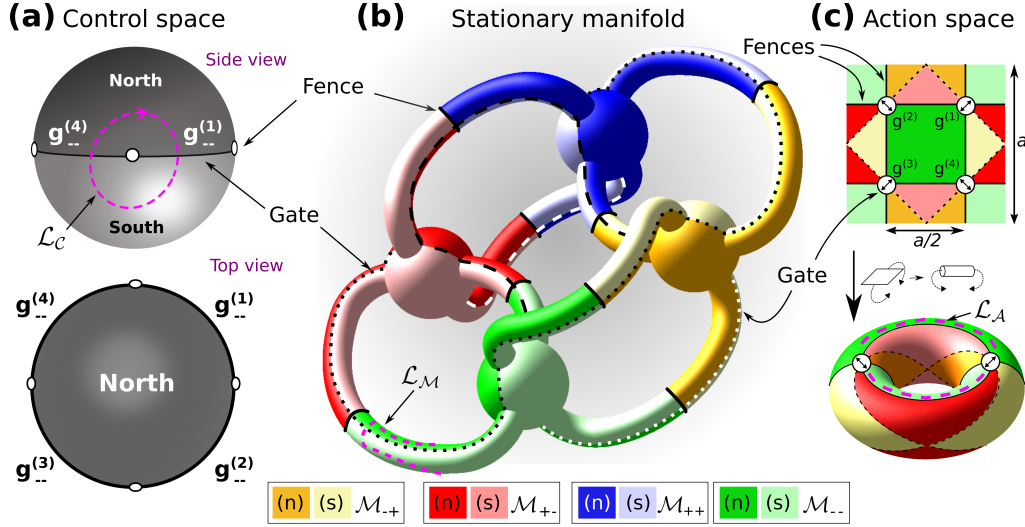


FIG. 2. Control space  $\mathcal{C}$  (a), the stationary surface  $\mathcal{M}$  (b), and action space  $\mathcal{A}$  (c). Each color in  $\mathcal{M}$  (b) represents a bijective area, as indicated (dark colors for northern areas and soft colors for southern areas). The solid lines are fences and the dotted and dashed lines are gates. In  $\mathcal{C}$  (a) the solid lines on the equator are the segments of minima of the gates and the empty circles the fences. In  $\mathcal{A}$  (c) the fences are represented by solid lines and the gates by circles with arrows indicating the possible transport directions. The color of  $\mathcal{A}$  is given by the projection of half of  $\mathcal{M}$  into  $\mathcal{A}$ . The violet dashed line in (a) is a control loop,  $\mathcal{L}_C = (g_1, \bar{g}_4)$ , that crosses two gates and induces colloidal transport. The preimage loop in  $\mathcal{M}_{--}$  is indicated by  $\mathcal{L}_M$  and the corresponding loop in  $\mathcal{A}$  by  $\mathcal{L}_A$ .

of four segments with identical signs of the eigenvalues form a full equator, see Fig. 2a.

The gates split  $\mathcal{C}$  in two parts, the south (s) and the north (n), see Fig. 2a. They also split each submanifold of  $\mathcal{M}$  in two parts  $\mathcal{M}_{\alpha\beta} = \mathcal{M}_{\alpha\beta}^{(n)} \cup \mathcal{M}_{\alpha\beta}^{(s)}$ , see Fig. 2b. This splitting is very convenient since the resulting regions  $\mathcal{M}_{\alpha\beta}^{(\nu)}$  with  $\nu = n, s$  are simply connected bijective areas. That is, there are no holes in  $\mathcal{M}_{\alpha\beta}^{(\nu)}$  and the correspondences between  $\mathcal{M}_{\alpha\beta}^{(\nu)}$  and the other spaces ( $\mathcal{C}$  and  $\mathcal{A}$ ) are unique.

### III. RESULTS

We are now in a position to understand the colloidal motion. Let  $\mathcal{L}_C$  be a closed modulation loop of the external field in  $\mathcal{C}$ .  $\mathcal{L}_C$  has four preimage loops in  $\mathcal{M}$ , one in each submanifold  $\mathcal{M}_{\alpha\beta}$ . Only the loop lying in  $\mathcal{M}_{--}$  is populated with colloids. This populated loop can be then projected into  $\mathcal{A}$  where we can read the actual trajectory of the colloids. Loops  $\mathcal{L}_C$  that induce colloidal transport from one unit cell to another in  $\mathcal{A}$  are only those that cross at least two different gates in control space, which is equivalent to enclosing at least one fence in  $\mathcal{C}$ . When  $\mathcal{L}_C$  crosses the segment  $g_{--}^{(i)}$  in  $\mathcal{C}$ , the corresponding loop that transport the colloids in  $\mathcal{A}$  also crosses the gate  $g^{(i)}$ . Each gate in  $\mathcal{C}$  can be crossed from the north to the south or from the south to the north, which in  $\mathcal{A}$  results in opposite senses. Let  $\mathcal{L}_C = (g_i, \bar{g}_j)$  be a loop of the external field that starts on the north of  $\mathcal{C}$ , then goes to the south

of  $\mathcal{C}$  crossing the segment of minima of the potential of the gate  $i$  ( $g_{--}^{(i)}$ ) and returns to the initial point in the north of  $\mathcal{C}$  using the segment of minima of the gate  $j$ . An example of such a loop is represented in Fig. 2a. The phase diagram of the colloidal motion in the  $g_i - \bar{g}_j$  plane is depicted in Fig. 3a. It has been obtained (i) theoretically by translating loops in  $\mathcal{C}$  into loops in  $\mathcal{A}$  using the stationary surface  $\mathcal{M}$  and (ii) with standard Brownian dynamics simulations. Details of the simulations are provided in the Appendix. The agreement between theory and simulations is perfect.

Loops that cross the same gate twice, i.e.,  $\mathcal{L}_C = (g_i, \bar{g}_i)$ , do not induce transport between different unit cells (the initial and the final positions are the same). Loops that cross different gates induce transport between nearest or second nearest unit cells. There are two possible routes for each of the nearest unit cells (see e.g.,  $\mathcal{L}_C = (g_1, \bar{g}_4)$  and  $(g_2, \bar{g}_3)$ ) and only one in the case of second nearest unit cells (e.g.,  $\mathcal{L}_C = (g_1, \bar{g}_3)$ ). In Fig. 3b we show Brownian dynamics trajectories for selected modulation loops.

The colloidal transport is very robust against internal and external perturbations. The shape of  $\mathcal{L}_C$ , for example, is completely irrelevant. Only the gates that  $\mathcal{L}_C$  crosses are important. In Fig. 4 we show the trajectories in action space for three modulation loops that cross the same two gates,  $\mathcal{L}_C = (g_1, \bar{g}_2)$ , yet following different paths. The trajectories in  $\mathcal{A}$  differ but the starting and ending allowed regions are the same. The motion is also robust against changes in the speed of the modulation, the thermal noise, and properties of the colloidal parti-

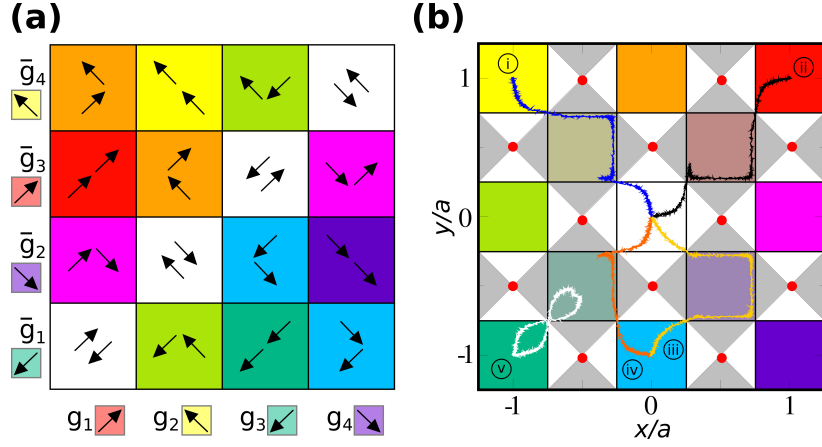


FIG. 3. (a) Phase diagram of the colloidal motion in the plane  $g_i - \bar{g}_j$  for the fundamental modulation loops in control space  $\mathcal{L}_C = (g_i, \bar{g}_j)$ . The loop starts in the north of  $\mathcal{C}$  then goes to the south using the gate segment  $g_{--}^{(i)}$  and returns to the south through the segment  $\bar{g}_{--}^{(j)}$ . Each color represents a transport direction. The arrows indicate which gates are crossed and in which sense. (b) Examples of the trajectories of the colloids in  $\mathcal{A}$  according to BD simulations for the modulation loops: (i)  $\mathcal{L}_C = (g_2, \bar{g}_4)$ , (ii)  $\mathcal{L}_C = (g_1, \bar{g}_3)$ , (iii)  $\mathcal{L}_C = (g_4, \bar{g}_1)$ , (iv)  $\mathcal{L}_C = (g_3, \bar{g}_2)$ , and (v)  $\mathcal{L}_C = (g_1, \bar{g}_1)$ . The solid lines are the fences in  $\mathcal{A}$ . The forbidden regions are marked with a middle red circle. The allowed regions are coloured according to the phase diagram in (a). We show four trajectories, (i) to (iv), corresponding to loops that induce colloidal transport (the initial position of the colloids is the allowed region centered at the origin), and one trajectory corresponding to a topologically trivial control loop (v) that does not induce transport (the initial position of the colloid is the allowed region centered at  $x/a = -1$  and  $y/a = -1$ ). The magnetic pattern is also represented using white and grey regions.

cles such as size, mass, effective susceptibility, etc (see an example in Fig. 5). Therefore we can transport in a dispersion-free and precise way a collection of particles with a broad distribution of masses, sizes, etc.

The reason behind this robustness is that the transport direction depends only on a topological invariant, and hence it is topologically protected. For each loop in  $\mathcal{M}$  we can define a set of 10 winding numbers, two for each hole of  $\mathcal{M}$ .  $S_{\mathcal{M}}$ , the set of winding numbers of the loop in  $\mathcal{M}_{--}$ , is the topological invariant. In each of the regions of the phase diagram  $S_{\mathcal{M}}$  does not vary. Alternatively we can define the topological invariant of loops in  $\mathcal{A}$  and  $\mathcal{C}$ . The loop that lies in  $\mathcal{M}_{--}$  is projected into a loop in  $\mathcal{A}$  and  $\mathcal{C}$ . Since  $\mathcal{M}_{--}$  is topologically equivalent to control space without the fences,  $\mathcal{C}'$ , the correspondence between loops in  $\mathcal{M}_{--}$  and  $\mathcal{C}'$  is bijective.  $S_{\mathcal{C}}$ , the set of winding numbers of loops around the fences in  $\mathcal{C}'$  induce corresponding winding numbers of loops around the torus in action space ( $S_{\mathcal{A}} = \{w_1, w_2\}$  with  $w_i = 0, \pm 1$ ) via the loops in  $\mathcal{M}_{--}$ . Each of the eight non-zero values of  $S_{\mathcal{A}}$  corresponds to a type of transport in  $\mathcal{A}$ .  $S_{\mathcal{A}}$  and  $S_{\mathcal{C}}$  are also topological invariants, they remain unchanged for each type of transport, i.e, in each region of the phase diagram of Fig. 3a.

How is it possible to change the direction of transport if it is topologically protected? There are always operations that break the topological protection. This is precisely what happens at the interface between two transport directions in the phase diagram, cf. Fig. 3a. At the interfaces between two different transport modes the topological protection is lost allowing for a change in the

transport mode. This occurs for modulation loops that cross at least one of the fences in control space. In Fig. 6 we show an example of this process. The loop labeled as (1) lies entirely on the north of  $\mathcal{C}$ . That is, it does not cross gates and hence does not induce transport between different unit cells. The corresponding loops in  $\mathcal{M}$  lie on the northern areas of  $\mathcal{M}$ . There is one loop in each of the submanifolds of  $\mathcal{M}$ . Fig. 6 shows only the loops in  $\mathcal{M}_{--}$  and  $\mathcal{M}_{+-}$ . When the loop in  $\mathcal{C}$  touches one of the fences (see loop (2) in Fig. 6) the loops in  $\mathcal{M}_{--}$  and  $\mathcal{M}_{+-}$  join at the fence (the loops in  $\mathcal{M}_{++}$  and  $\mathcal{M}_{-+}$  also join at a different fence). At this point the colloids, which follow the loop in  $\mathcal{M}_{--}$ , have two alternative paths: (i) a loop that resides entirely in the north of  $\mathcal{M}_{--}$  and (ii) a loop that lies in both the north and the south of  $\mathcal{M}_{--}$  and hence induce colloidal transport between different cells. The motion is not topologically protected in the sense that two different trajectories are possible. Next, we expand the loop in  $\mathcal{C}$  such that it encloses one fence in  $\mathcal{C}$  and hence crosses two gates, see loop (3) in Fig. 6. In  $\mathcal{M}$  the loops in  $\mathcal{M}_{--}$  and  $\mathcal{M}_{+-}$  are now disjointed and have interchanged a segment at the fence. The result is two loops that no longer reside in the northern areas of  $\mathcal{M}$ . The loop in  $\mathcal{M}_{--}$  winds around the holes of  $\mathcal{M}$  inducing colloidal transport. The direction of transport has changed with respect to the initial loop (1).

Due to the thermal noise in Brownian dynamics simulations the particles fluctuate around the minima of the potential, exploring the neighborhood of  $\mathcal{M}_{--}$  in  $\mathcal{C} \otimes \mathcal{A}$ . Hence, modulation loops in control space that do not cross a fence, but pass close enough to it, might also be



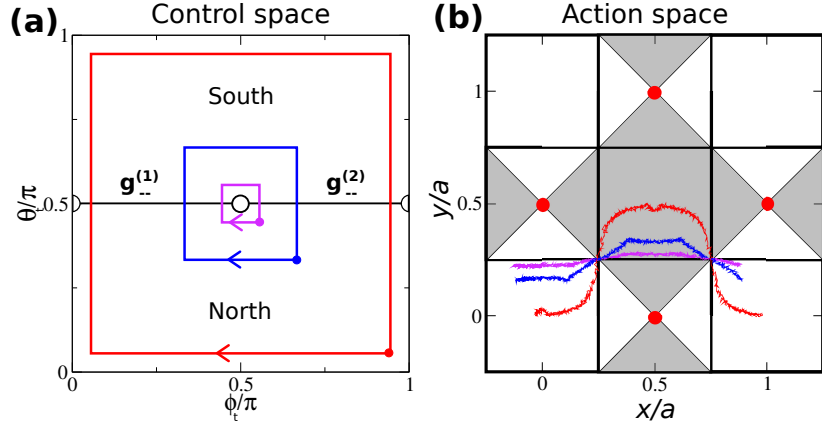


FIG. 4. (a) Modulations loops in control space of type  $\mathcal{L}_C = (g_1, \bar{g}_2)$ . The direction and the starting point of the loops are indicated by arrows and filled circles, respectively. The empty circles are the fences in control space and the horizontal black line are the gates as indicated. (b) Trajectories in action space corresponding to the loops showed in (a). The trajectories are coloured according to the color of the loops in (a). The white and grey areas indicate the magnetic pattern. The squares are the allowed and forbidden areas of action space. The forbidden areas are highlighted with a red circle in the middle. The initial position of the colloids is the allowed area centered at the origin.

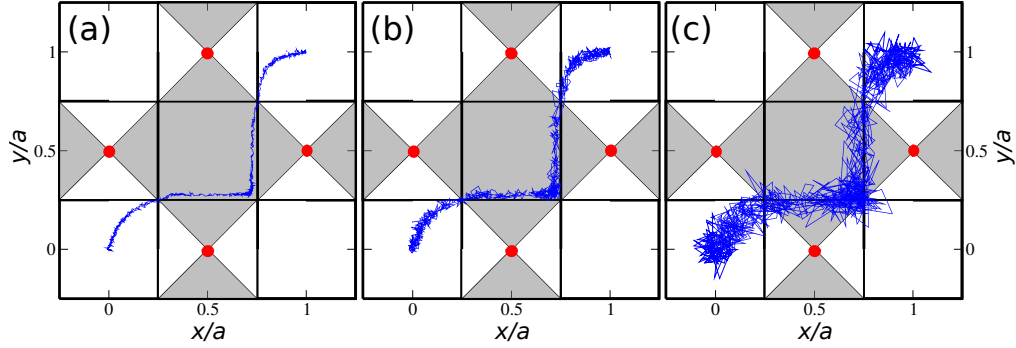


FIG. 5. Trajectories in action space of a diamagnetic colloid under a control loop  $\mathcal{L}_C = (g_1, \bar{g}_3)$  for different values of the scaled temperature  $k_B T / \epsilon = 0.01$  (a),  $0.1$  (b), and  $1.0$  (c). Here  $k_B$  is the Boltzmann constant, and  $\epsilon$  sets the unit of energy of the magnetic potential  $V$ . The white and grey areas indicate the magnetic pattern. The squares are the allowed and forbidden areas of action space. The forbidden areas are highlighted with a red circle in the middle. The initial position of the colloids is the allowed area centered at the origin.

topologically unprotected, leading to two differing transport modes in  $\mathcal{A}$ . How close the control loop has to be to the fence in order to be deprotected depends on the magnitude of the thermal noise. The thermal noise effectively expand the fences in  $\mathcal{C}$  into the surrounding areas, and broaden the topological transition in  $\mathcal{A}$ .

#### IV. DISCUSSION

We have explained the motion of diamagnetic colloids for which the effective susceptibility is negative. Paramagnetic colloids have a positive effective susceptibility, and hence will follow the maxima of  $V$ . The minima and the maxima of  $V$  always comove in  $\mathcal{A}$  separated by  $\mathbf{r} = (a/2, a/2)$ . Therefore, paramagnetic colloids perform the same motion as diamagnets but displaced by  $\mathbf{r}$ .

From a experimental view point, it is possible to use

magnetic bubble lattices [14] or lithographic patterns [15] to generate the pattern. Possible methods to levitate the colloids above the pattern consist on using a ferrofluid solvent [13] and the deposition of a polymer layer [16] on the magnetic pattern.

The colloidal transport is fully determined by the topology of the manifold  $\mathcal{M}$ , which is unique for each type of magnetic pattern. For example, the stationary manifold of a hexagonal pattern is a genus 7 surface [13]. There, the modulation loops in  $\mathcal{C}$  that induce transport of colloids must cross the fences in  $\mathcal{C}$ , which are lines instead of points as in the present study. As a result, transport modes of hexagonal and square patterns are completely different. In both, hexagonal and square lattices, the topological invariant in  $\mathcal{A}$  is the set of two winding numbers around the hole in  $\mathcal{A}$ . This is just a consequence of the dimension of  $\mathcal{A}$ . Control space  $\mathcal{C}$  neither contains all the information. For example, in square

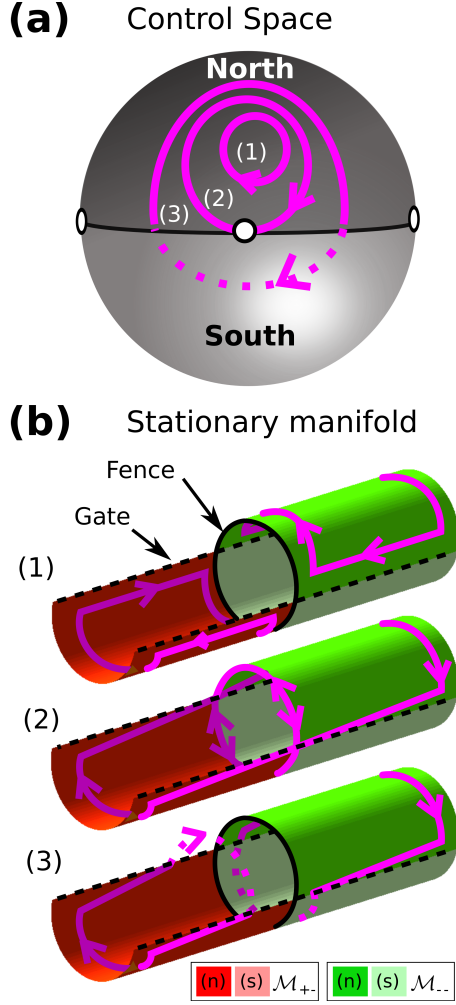


FIG. 6. Control space  $\mathcal{C}$  (a) and part of the stationary manifold  $\mathcal{M}$  (b). Three modulation loops  $\mathcal{L}_{\mathcal{C}}$  in control space and their corresponding loops in  $\mathcal{M}_{--}$  and  $\mathcal{M}_{+-}$  are indicated by violet lines. The loops are represented with a solid (dashed) line in the north (south) of control space and the stationary manifold. The arrows indicate the direction of the loops.

lattices the transition between transport modes occurs for those loops that cross a fence. However, in hexagonal lattices, a fence crossing loop in  $\mathcal{C}$  is a necessary but not sufficient condition to change the transport mode. What fully determines the transport modes is the stationary manifold  $\mathcal{M}$  (the topology, the fences, and how  $\mathcal{M}$  is projected into  $\mathcal{A}$  and  $\mathcal{C}$ ). In  $\mathcal{M}$  the topological invariant is the set of winding numbers around the holes, which is very different in square ( $\mathcal{M}$  has genus 5) and hexagonal ( $\mathcal{M}$  has genus 7) lattices.

The topologically protected transport modes we have shown here can be understood as bulk modes sustained (driven) by an external field. The transport occurs in the bulk of the periodic system. Other forms of topologically protected motion occur at the edges of a periodic system, such as e.g., the motion of electrons in topological insulators [17], mechanical solitons [18–20], phonons [21],

and photons [22, 23] among others. There, a perturbation populates an edge state that cannot scatter into the bulk due to the topology of the system. Our theory is transferable to other systems with the same symmetry. Hence, topological bulk states might exist in e.g. excitons in superlattices [24, 25], tight-binding models [26], and cold atoms in optical lattices [27]. Topologically protected edge states might also occur at the borders of finite magnetic lattices. Their topological properties might be substantially different from those of bulk states. How the edge states in our particle system compare to other edge states in wave systems is a very interesting subject for future studies.

In wave systems, such as e.g. topological insulators, the topology of the band structure is characterized by the Chern numbers of the bands. Each Chern number can be computed as an integral over the Berry curvature of the band [28]. In our particle system we describe the topological protection in terms of the stationary manifold. Both descriptions are probably equivalent in some form.

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## Appendix A: Brownian Dynamics Simulations

We use Brownian Dynamics to simulate the motion of a diamagnetic colloid above the pattern. The coordinates in action space are  $\mathbf{x}_{\mathcal{A}}$ , and the equation of motion is given by

$$\xi \frac{d\mathbf{x}_{\mathcal{A}}(t)}{dt} = -\nabla_{\mathcal{A}} V(\mathbf{x}_{\mathcal{A}}, \mathbf{H}_{\text{ext}}(t)) + \boldsymbol{\eta}(t),$$

where  $t$  is the time,  $\xi$  is the friction coefficient, and  $\boldsymbol{\eta}$  is a Gaussian random force with a variance given by the fluctuation-dissipation theorem. The magnetic potential  $V$  has contributions from the external field  $\mathbf{H}_{\text{ext}}$  and the magnetic pattern (see the main text).

The equation of motion is integrated in time with a standard Euler algorithm:

$$\mathbf{x}_{\mathcal{A}}(t + \Delta t) = \mathbf{x}_{\mathcal{A}}(t) - \nabla_{\mathcal{A}} V \Delta t + \boldsymbol{\delta r}, \quad (\text{A1})$$

where  $\Delta t$  is the time step, and  $\boldsymbol{\delta r}$  is a random displacement sampled from a gaussian distribution with standard deviation  $\sqrt{2\Delta t k_B T / \xi}$ . Here  $k_B$  is the Boltzmann constant, and  $T$  is absolute temperature. Before starting the modulation loop in  $\mathcal{C}$  we first equilibrate the system by running  $10^4$  time steps such that the colloids find the minimum of the magnetic potential at  $t = 0$ .

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